

Design and Performance Assessment of Robust Restricted Structure Optimal Control Systems

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ABSTRACT

It is evident that military applications for 21st century will be highly complex, multivariable systems. Designing optimal controllers for such applications will require the use of mathematical models which describe the complexity of the underlying processes as accurately as possible. For robustness it is essential that these models include the uncertainties in the estimated process dynamics and trajectories. Controllers resulting from optimal control strategies (like LQG, H_2 , H_∞) will usually be of very high order and that can cause implementation and computational problems. Some of these problems can be overcome by using controllers that are of lower order and restricted structure. However, the design and/or tuning of such controllers in order to provide the performance comparable with full-order solutions is still a very contentious issue. Added to this is the need to achieve robust properties and performance specifications required by military applications. Very few methods for designing restricted-structure controllers exist that allow the performance and robustness objectives to be combined into one relatively simple optimisation problem. This lecture presents an LQG/ H_2 -based method that tackles the above mentioned issues. The LQG/ H_2 -based criterion is minimized in such a way that the resulting controller is of the desired form and is causal. A simple analytic solution cannot be obtained, however a straightforward direct optimization problem can be established which provides the desired solution. From this solution it is also possible to derive a tool for assessing the performance of existing controllers.

1.0 INTRODUCTION

The design and performance assessment of optimal controllers (Kwakernaak and Sivan, 1972 [1], Desborough and Harris 1993 [29], Grimbale and Uduchi 2001 [26]) is considered as well as the ways in which robustness properties can be modified and assessed. The minimization of quadratic cost functions for stochastic linear systems in such a way that the robustness margins are improved is of interest. There are of course well known guaranteed robustness properties for systems with state feedback (Anderson and Moore 1971 [2], Safonov and Athans 1986 [3], Doyle 1978 [4]) and in some problems these properties can be recovered using a loop transfer recovery approach ([5]-[10]). However, more direct methods of tuning robustness properties would be valuable (Doyle and Stein, 1981 [11]). This is explored for both full-order optimal output feedback controllers and for low order (restricted structure) optimal solutions.

This paper also examines the important issue of performance assessment of existing controllers with regard to robustness and other steady state properties. In recent years there has been a lot of research devoted to

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assessing the performance of existing control systems with the objective of determining if some degrees of freedom exist to optimise such control systems. The idea is to compare the performance of the existing control against some known reference standard and produce an index ranging between 0 and 1, which indicates how well the control system is performing. Desborough and Harris (1993 [29]) considered the assessment of control loop performance for both feedback and feedforward control using minimum variance as the benchmark cost measure. Huang and Shah (1999 [30]) summarised the state of the art in a monograph, mostly focussing on the minimum variance cost index as the performance assessment measure. By reducing the variance of the output deviation systems can be operated as close to their physical constraints as possible. However, due to the problems in implementing minimum variance controllers, (McGregor and Tidwell 1977 [31], wide bandwidth, large noise amplification) and because the minimum variance controllers tend to be of high order relative to classical designs (such as PID), the issues of performance assessment were extended to include LQG, GPC and GMV criteria for restricted structure reduced order controllers (Grimble [32], Ordys et al [28], Uduehi et al [27]). However, none of the performance assessment indices developed so far tackle the issues of robustness.

Two aspects of the design of full-order and restricted structure optimal controllers are considered in the following. The first involves robustness improvement and the second is concerned with the noise rejection properties. An LQG criterion is to be optimized but the usual robustness and noise rejection properties will not hold if the controller structure is limited to say a PID or a low order lead-lag form. Grimble (1999 [12], 2000 [13]) introduced a polynomial systems approach to restricted structure optimal control design and this is the philosophy followed here. However, the design of such controllers previously focussed on performance issues and the robustness/noise-rejection aspects were not considered in any detail. The strategy for robustness improvement is to add a fictitious signal and a sensitivity costing term in the criterion. This enables the penalty on sensitivity to be directed at modifying the robustness properties. The normal LQG cost-index does of course include error and control signals that depend upon the sensitivity functions. However, this does not enable these sensitivity terms to be costed in a particular way. The proposed robustness weighting term gives free choice of the weighting function and enables the H_2 norm of a weighted sensitivity function to be minimized. This relates to the definition of a so-called Dual Criterion (Grimble, 1986 [14]), but the results here are focussed on the design issues and they use what might be termed a Kucera polynomial systems approach, (1980 [15]). The impact of a coloured measurement noise model on the controller design and on the robustness and performance properties is examined. The frequency-domain polynomial systems approach is particularly helpful when determining the frequency response behaviour to noise and disturbance signals (Grimble, 1994 [16]). The questions to be answered are how the sensitivity/robustness weighting and the measurement noise models can be selected to optimize the robustness and noise rejection properties. The impact of such a weighting on both the full-order optimal and on the restricted structure controllers will be explored.

2.0 SYSTEM MODEL

The system shown in Fig. 1 is assumed to be linear, continuous-time and single-input, single-output. The external white noise sources drive colouring filters which represent the reference $W_r(s)$, measurement noise $W_n(s)$, robustness modification $W_p(s)$ and disturbance $W_d(s)$ subsystems. The robustness model does not exist physically but is introduced for design modification. The system equations become:

Input disturbance:
$$d(s) = W_d(s)\xi(s) \quad (2.1)$$

Robustness signal:
$$p(s) = W_p(s)\eta(s) \quad (2.2)$$

Output: $y(s) = d(s) + p(s) + W(s)u(s)$ (2.3)

Reference: $r(s) = W_r(s)\zeta(s)$ (2.4)

Tracking error: $e(s) = r(s) - y(s)$ (2.5)

Observations: $z(s) = y(s) + n(s)$ (2.6)

Measurement noise: $n(s) = W_n(s)\omega(s)$ (2.7)

Control signal: $u(s) = C_0(s)(r(s) - z(s))$ (2.8)

The system transfer functions are all assumed to be functions of the Laplace transform complex number in the complex frequency domain. For notational simplicity the arguments in $W(s)$, and the other models are omitted.

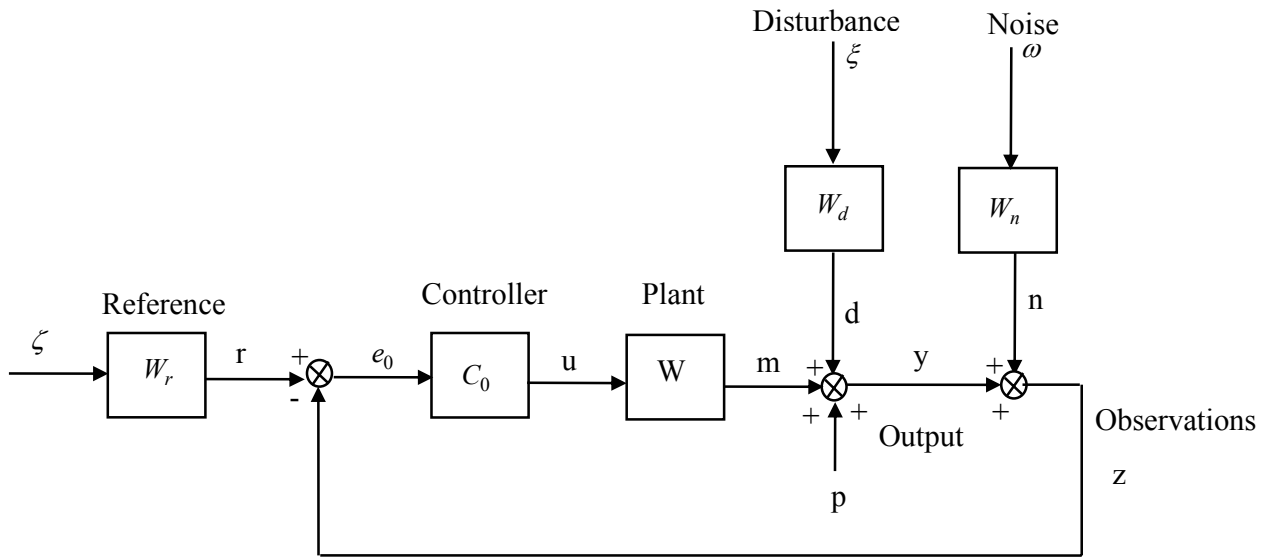


Figure 1: Single Degree of Freedom Unity Feedback Control System with Measurement Noise, Reference and Disturbance Models.

Assumptions

1. The white noise sources, ξ , ω , η and ζ are zero-mean and mutually statistically independent. The intensities of these signals are without loss of generality taken to be of value unity.
2. The system W is assumed free of unstable hidden modes and the reference W_r , noise W_n , robustness W_p and disturbance W_d subsystems are asymptotically stable.

The following expressions may easily be derived for the output, error, observations, controller input, control and sensitivity costing signals:

Output: $y(s) = WC_0(1 + WC_0)^{-1}(r(s) - n(s)) + (1 + WC_0)^{-1}(d(s) + p(s))$ (2.9)

Error: $e(s) = r(s) - y(s) = (1 + WC_0)^{-1}(r(s) - d(s) - p(s)) + WC_0(1 + WC_0)^{-1}n(s)$ (2.10)

Observations: $z(s) = WC_0(1 + WC_0)^{-1}r(s) + (1 + WC_0)^{-1}(d(s) + n(s) + p(s))$ (2.11)

Controller input: $e_0(s) = r(s) - z(s) = (1 + WC_0)^{-1}(r(s) - d(s) - n(s) - p(s))$ (2.12)

Control signal: $u(s) = (1 + WC_0)^{-1}C_0(r(s) - d(s) - n(s) - p(s))$ (2.13)

Robustness signal: $h(s) = (1 + WC_0)^{-1}p(s)$ (2.14)

These equations include the following sensitivity operators:

Sensitivity: $S = (I + WC_0)^{-1}$ (2.15)

Complementary sensitivity: $T = I - S = WC_0S$ (2.16)

Control sensitivity: $M = C_0S = C_0(I + WC_0)^{-1}$ (2.17)

The system shown in Fig. 1 may be represented in polynomial form (Kailath 1980 [22]), where the system transfer functions are written as:

System: $W = A^{-1}B$ (2.18)

Reference generator: $W_r = A^{-1}E$ (2.19)

Input disturbance: $W_d = A^{-1}C_d$ (2.20)

Measurement noise: $W_n = A^{-1}C_n$ (2.21)

Robustness signal model: $W_p = A^{-1}C_p$ (2.22)

There is no loss of generality in assuming these models have a common denominator A polynomial. The various polynomials are not necessarily coprime but the system transfer function is assumed to be free of unstable hidden modes. The coprime representation of the system is denoted by $A_0^{-1}B_0$, where $B=B_0U_0$ and $A=A_0U_0$.

The spectrum of the signal $r(s)-d(s)-n(s)-p(s)$ in equations (2.12) and (2.13) is denoted by $\Phi_{ff}(s)$ and a generalised spectral-factor Y_f may be defined from this spectrum, using:

$$Y_f Y_f^* = \Phi_{ff} = \Phi_{rr} + \Phi_{dd} + \Phi_{nn} + \Phi_{pp} \quad (2.23)$$

In polynomial form $Y_f = A^{-1}D_f$. The disturbance model is assumed to be such that D_f is strictly Hurwitz and satisfies:

$$D_f D_f^* = EE^* + C_d C_d^* + C_n C_n^* + C_p C_p^* \quad (2.24)$$

The role of the robustness model W_p may now be explained since it is one of the components in the combined signal spectrum Φ_{ff} defined in (2.23). Clearly, if Φ_{pp} is dominant then the spectrum $\Phi_{ff} \rightarrow \Phi_{pp}$.

This implies that $\Phi_{e_0 e_0} \rightarrow S \Phi_{pp} S^*$ and this spectrum will provide the robustness term (suitably weighted) introduced in the cost function, in the next section.

3.0 LQG CRITERION AND RESTRICTED STRUCTURE CONTROL PROBLEM

The LQG cost-function to be minimized (*Youla et al* 1976 [17]) is defined as:

$$J = \frac{1}{2\pi} \oint_D \{Q_c(s) \Phi_{ee}(s) + R_c(s) \Phi_{uu}(s) + P_c(s) \Phi_{e_0 e_0}(s)\} ds \quad (3.1)$$

where Q_c , R_c , P_c represent dynamic weighting elements, acting on the spectra of the error $e(t)$, feedback control $u(t)$ and controller input $e_0(t)$ signals. The R_c weighting term is assumed to be positive definite and Q_c , P_c are assumed to be positive-semidefinite on the D contour of the s-plane. The robustness weighting term can be motivated as in the dual criterion results of *Grimble* (1986 [14]). The error, control and robustness weightings can be written in polynomial form as:

$$Q_c = \frac{Q_{cn}}{A_q^* A_q} = \frac{B_q^* B_q}{A_q^* A_q}, \quad R_c = \frac{R_{cn}}{A_r^* A_r} = \frac{B_r^* B_r}{A_r^* A_r} \quad \text{and} \quad P_c = \frac{P_{cn}}{A_q^* A_q} = \frac{B_p^* B_p}{A_q^* A_q} \quad (3.2)$$

where A_q is a Hurwitz polynomial and A_r is a strictly Hurwitz polynomial (*Grimble and Johnson* 1988 [19]). The problem will be to minimize the above criterion with the controller chosen to have a specified structure [13]. The P_c weighting term represents a robustness weighting to be explained later. For later use let the weightings be rewritten using the common denominator, so that:

$$\tilde{Q}_c = Q_c + P_c = \frac{Q_{cn} + P_{cn}}{A_q^* A_q} = \frac{\bar{Q}_{cn}}{A_q^* A_q} \quad \text{or} \quad \tilde{Q}_c = \tilde{Q}_{cn} / (A_q A_q^* A_r A_r^*) = \tilde{Q}_{cn} / (A_w A_w^*)$$

where

$$\tilde{Q}_{cn} = \bar{Q}_{cn} A_r A_r^* \quad \text{and} \quad A_w = A_q A_r. \quad \text{Similarly for the control weighting:}$$

$$\tilde{R}_c = \frac{R_{cn}}{A_r^* A_r} = \tilde{R}_{cn} / (A_q A_q^* A_r A_r^*) = \tilde{R}_{cn} / (A_w A_w^*)$$

where $\tilde{R}_{cn} = R_{cn} A_q A_q^*$.

Theorem 3.1: Restricted Structure Single Degree of Freedom LQG Control Problem

Consider the LQG error and control weighted criterion defined in (3.1), and the system introduced in §2. The conditions that determine the LQG controller of restricted structure are derived below. The derivation includes the robustness/sensitivity costing and coloured measurement noise model. These terms were not considered in the previous polynomial approaches to the restricted structure control design problem. The cost-function to be minimised was defined in as,

$$J = \frac{1}{2\pi} \oint_D \{Q_c \Phi_{ee} + R_c \Phi_{uu} + P_c \Phi_{e_0 e_0}\} ds$$

Noting the independence of the noise sources and recalling (2.10) to (2.13) and the definitions for sensitivity in equations (2.15) to (2.17) obtain by substituting in (3.1):

$$J = \frac{1}{2\pi j_D} \oint \left\{ \begin{aligned} &Q_c(1-WM)\Phi_{ff}(1-W^*M^*) + P_c(1-WM)\Phi_{ff}(1-M^*W^*) \\ &+ R_c M \Phi_{ff} M^* - Q_c(1-WM)\Phi_{nn} - \Phi_{nn}(1-M^*W^*)Q_c + Q_c \Phi_{nn} \end{aligned} \right\} ds$$

$$J = \frac{1}{2\pi j_D} \oint \left\{ \begin{aligned} &(W^* \tilde{Q}_c W + R_c) M \Phi_{ff} M^* - M^* W^* (\tilde{Q}_c \Phi_{ff} - Q_c \Phi_{nn}) \\ &-(\tilde{Q}_c \Phi_{ff} - Q_c \Phi_{nn}) W M + \tilde{Q}_c \Phi_{ff} - Q_c \Phi_{nn} \end{aligned} \right\} ds \quad (3.3)$$

where $\tilde{Q}_c = Q_c + P_c$ and this may be written in the alternative forms $\tilde{Q}_c = \frac{\bar{Q}_{cn}}{A_q A_q^*} = \frac{Q_{cn} + P_{cn}}{A_q A_q^*}$ or using a *common denominator* for the spectral factor:

$$\tilde{Q}_c = \tilde{Q}_{cn} / (A_q A_q^* A_r A_r^*) = \tilde{Q}_{cn} / (A_w A_w^*) \quad (3.4)$$

where $\tilde{Q}_{cn} = \bar{Q}_{cn} A_r A_r^*$. Then define,

$$\Phi_{fp} = W^* (\tilde{Q}_c \Phi_{ff} - Q_c \Phi_{nn}) \quad (3.5)$$

and $\Phi_{ff} = \Phi_{rr} + \Phi_{dd} + \Phi_{nn} + \Phi_{pp}$.

The *generalised spectral factors* Y_c and Y_f due to Shaked (1976 [21]) may now be defined, using:

$$Y_c^* Y_c = W^* \tilde{Q}_c W + R_c \quad (3.6)$$

$$Y_f Y_f^* = \Phi_{ff} = \Phi_{rr} + \Phi_{dd} + \Phi_{nn} + \Phi_{pp} \quad (3.7)$$

Completing the squares in equation (3.3) obtain:

$$J = \frac{1}{2\pi j_D} \oint \left\{ (Y_c M Y_f - \frac{\Phi_{fp}}{Y_c^* Y_f^*}) (Y_c M Y_f - \frac{\Phi_{fp}}{Y_c^* Y_f^*})^* + \Phi_0 \right\} ds \quad (3.8)$$

where

$$\Phi_0 = \tilde{Q}_c \Phi_{ff} - Q_c \Phi_{nn} - \frac{\Phi_{fp} \Phi_{fp}^*}{Y_c Y_c^* Y_f Y_f^*} \quad (3.9)$$

Substituting in the spectral-factor expressions (3.6) and (3.7), using the polynomial system models in equations (2.18) to (2.22), obtain

$$Y_f Y_f^* = (E E^* + C_d C_d^* + C_n C_n^* + C_p C_p^*) / (A A^*) \quad (3.10)$$

$$Y_c^* Y_c = (B^* \tilde{Q}_{cn} B + A^* \tilde{R}_{cn} A) / (A^* A_w^* A_w A) \quad (3.11)$$

where

$$\tilde{Q}_{cn} = (Q_{cn} + P_{cn})A_r A_r^* \text{ and } \tilde{R}_{cn} = R_{cn} A_q A_q^* \quad (3.12)$$

To obtain the polynomial spectral-factors define the filter D_f and control D_c spectral factors from (3.6) and (3.7) respectively, as:

$$D_f D_f^* = E E^* + C_d C_d^* + C_n C_n^* + C_p C_p^* \quad (3.13)$$

$$D_c^* D_c = B^* \tilde{Q}_{cn} B + A^* \tilde{R}_{cn} A \quad (3.14)$$

Recalling that A_q and A_r are normally chosen to be coprime, the generalized spectral factors may be written in the form:

$$Y_f = A^{-1} D_f \text{ and } Y_c = A_c^{-1} D_c \text{ where } A_c = A A_w \text{ and } A_w = A_q A_r$$

The various terms in the criterion (3.1) may now be simplified by substituting from the polynomial system models in §2 and the spectral factor results given above.

$$\begin{aligned} \Phi_{fp} &= W^* (\tilde{Q}_c \Phi_{ff} - Q_c \Phi_{nn}) = \frac{B^*}{A^*} \left(\frac{\tilde{Q}_{cn} D_f D_f^* - Q_{cn} C_n C_n^*}{A_q A_q^* A A^*} \right) \\ \Phi_{fp} / (Y_c^* Y_f^*) &= B^* \left(\frac{\tilde{Q}_{cn} D_f D_f^*}{A A_q} - \frac{Q_{cn} C_n C_n^*}{A A_q} \right) \frac{A_r^*}{D_c^* D_f^*} \end{aligned} \quad (3.15)$$

The measurement noise subsystem must be asymptotically stable and after cancellation of common terms may be written as: $A^{-1} C_n = A_{n0}^{-1} C_{n0}$, where $A = A_{n0} A_0$. The following diophantine equations must be introduced:

Feedback diophantine equations:

Calculate (G_0, H_0, F_0) with F_0 of minimum degree:

$$D_c^* G_0 + F_0 A A_q = B^* \tilde{Q}_{cn} A_r^* D_f \quad (3.16)$$

$$D_c^* H_0 - F_0 B A_r = A^* R_{cn} A_q^* D_f \quad (3.17)$$

Implied equation:

Multiplying (3.16) by $B A_r$ and (3.17) by $A A_q$ and adding the equations obtain:

$$D_c^* (G_0 B A_r + H_0 A A_q) = (B^* \tilde{Q}_{cn} B + A^* \tilde{R}_{cn} A) D_f$$

and after division by D_c^* obtain:

$$G_0 B A_r + H_0 A A_q = D_c D_f \quad (3.18)$$

Measurement noise equation:

$$D_c^* D_f^* X_0 + Y_0 A_{n0} A_q = B^* Q_{cn} C_{n0} C_n^* A_r^* \quad (3.19)$$

The terms in the cost-optimization problem may now be considered, substituting from the above polynomial equations. Substituting from the diophantine equations (3.16) and (3.17):

$$\frac{\Phi_{fp}}{Y_c^* Y_f^*} = \left(-\frac{G_0}{AA_q} + \frac{F_0}{D_c^*} \right) - \left(\frac{X_0}{A_{n0} A_q} + \frac{Y_0}{D_c^* D_f^*} \right) \quad (3.20)$$

Considering now the term $Y_c M Y_f$, writing $C_0 = C_{0d}^{-1} C_{0n}$, obtain :

$$Y_c M Y_f = \frac{D_c D_f C_{0n}}{AA_w (AC_{0d} + BC_{0n})} \quad (3.21)$$

The first squared term in (3.8), using (3.21) now becomes:

$$Y_c M Y_f - \frac{\Phi_{fp}}{Y_c^* Y_f^*} = \frac{[D_c D_f C_{0n} - (G_0 - X_0 A_0) A_r (AC_{0d} + BC_{0n})]}{AA_w (AC_{0d} + BC_{0n})} + \frac{(Y_0 - F_0 D_f^*)}{D_c^* D_f^*} \quad (3.22)$$

Substituting from the *implied diophantine* equation:

$$Y_c M Y_f - \frac{\Phi_{fp}}{Y_c^* Y_f^*} = \frac{[(H_0 A_{n0} A_q + X_0 A_r B) C_{0n} - (G_0 A_{n0} - X_0 A) A_r C_{0d}]}{A_{n0} A_w (AC_{0d} + BC_{0n})} + \frac{(Y_0 - F_0 D_f^*)}{D_c^* D_f^*} \quad (3.23)$$

This cost term expression may be written in the form:

$$Y_c M Y_f - \frac{\Phi_{fp}}{Y_c^* Y_f^*} = T_1^+ + T_1^- \quad (3.24)$$

where the term within the square brackets in (3.23) denoted by T_1^+ . This term is stable, since A_w is Hurwitz and the closed-loop characteristic polynomial $\rho_c = (AC_{0d} + BC_{0n})$ is required to be strictly Hurwitz for $J_{min} < \infty$. The final term in (3.23) is strictly unstable since D_c^* is strictly non-Hurwitz.

4.0 COST FUNCTION MINIMIZATION AND PARAMETRIC OPTIMIZATION PROBLEM

Given the simplification of terms in the cost-function presented above the cost minimization procedure may be followed (*Grimble and Johnson*, 1988 [19]). Note that the cost- function (3.8) may be written, using (3.24) as:

$$J = \frac{1}{2\pi j} \oint_D \{ (T_1^+ + T_1^-) (T_1^+ + T_1^-)^* + \Phi_0 \} ds \quad (4.1)$$

From the Residue theorem the integrals of the cross-terms $T_l^+ T_l^{-*}$, $T_l^- T_l^{+*}$ can be shown to be zero. This result follows because $\oint T_l^+ T_l^{-*} ds = - \oint T_l^- T_l^{+*} ds$ but the term $T_l^- T_l^{+*}$ is analytic for *all* s in the left half plane so that the sum of the residues obtained in calculating $\oint T_l^- T_l^{+*} ds$ is zero.

Note that this result still applies if the function $T_l^- T_l^{+*}$ contains poles on the $j\omega$ axis, since they can be avoided by the D contour, using small semi-circular detours in the left-half plane. These semi-circles are centred on these poles and do not contribute in the limiting case as the radius tends to zero. Also observe that the term containing $T_l^- T_l^{+*}$ could lead to an infinite cost should such terms be present. However, these may not be present, since the optimal control may be chosen so that they cancel. The practical case when this arises is when the error weighting includes an integrator $A_q(s) = s$. When the controller denominator $C_{0d}(s)$ includes integral action the A_q polynomial cancels throughout the term. The consequence is that the criterion can have a finite minimum, even though certain cost function terms include j axis poles. The cost-function therefore simplifies as:

$$J = \frac{1}{2\pi j_D} \oint \{ (T_l^+ T_l^{+*} + T_l^- T_l^{-*}) + \Phi_0 \} ds \quad (4.2)$$

Since the terms T_l^- and Φ_0 are independent of the controller, the criterion J is minimised when the first term involving T_l^+ is minimized. However, if the feedback controller C_0 has a restricted structure then it is unlikely that T_l^+ can be set to zero. It follows that to minimize the cost-function the first term in (4.2) should be minimised, through the choice of C_0 , namely:

$$J_0 = \frac{1}{2\pi j_D} \oint \{ T_l^+ T_l^{+*} \} ds \quad (4.3)$$

For a finite solution to this cost minimization problem to exist the T_l^+ term must be asymptotically stable. Inspection of this term:

$$T_l^+ = \frac{[(H_0 A_{n0} A_q + X_0 A_r B) C_{0n} - (G_0 A_{n0} - X_0 A) A_r C_{0d}]}{A_{n0} A_w (A C_{0d} + B C_{0n})} \quad (4.4)$$

reveals that all terms are asymptotically stable but the weighting A_q could include a j axis zero (A_q is only assumed to be Hurwitz). However, it is assumed that although the structure of the controller $C_0 = C_{0n} C_{0d}^{-1}$ is limited, C_{0d} will have zeros at the j axis zeros of the chosen weighting A_q . Thus, such a zero will cancel and under the given assumptions T_l^+ is asymptotically stable.

Then the LQG controller of restricted structure may be calculated from a simple direct optimization problem. First compute the filtering and control spectral factors D_f and D_c (strictly Hurwitz due to the system description) using:

$$D_f D_f^* = E E^* + C_d C_d^* + C_n C_n^* + C_p C_p^* \quad (4.5)$$

$$D_c^* D_c = B^* \tilde{Q}_{cn} B + A^* \tilde{R}_{cn} A \quad (4.6)$$

The following *regulating diophantine equations* must then be solved for (G_0, H_0, F_0) , with F_0 of minimum degree:

$$D_c^* G_0 + F_0 A A_q = B^* \bar{Q}_{cn} A_r^* D_f \quad (4.7)$$

$$D_c^* H_0 - F_0 B A_r = A^* R_{cn} A_q^* D_f \quad (4.8)$$

and the following *measurement noise diophantine equation* must be solved for (X_0, Y_0) , with Y_0 of smallest degree:

$$D_c^* D_f^* X_0 + Y_0 A_{n0} A_q = B^* Q_{cn} C_{n0} C_n^* A_r^* \quad (4.9)$$

The optimal controller $C_0 = C_{0n} C_{0d}^{-1}$ must then be found to minimize the following component in the cost-function term:

$$J_0 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \{T_1^+(j\omega) T_1^+(-j\omega)\} d\omega \quad (4.10)$$

where

$$T_1^+ = \frac{[(H_0 A_{n0} A_q + X_0 A_r B) C_{0n} - (G_0 A_{n0} - X_0 A) A_r C_{0d}]}{A_{n0} A_q A_r (A C_{0d} + B C_{0n})}$$

If the controller has a specified limited structure the minimum of the cost term J_{0min} will be non-zero. For an unconstrained solution the minimum is achieved when $T_1^+ = 0$ and the minimum of J_0 (denoted J_{0min}) is zero. It has been explained in Theorem 3.1, that the computation of the optimal feedback controller C_0 reduces to minimization of the term J_0 .

It is clear from (4.4) that T_1^+ can be written in the form:

$$T_1^+ = (C_{0n} L_1 - C_{0d} L_2) / (C_{0n} L_3 + C_{0d} L_4) \quad (4.11)$$

where $C_0 = C_{0n} / C_{0d}$ has a specified structure which can be as expressed below:

Reduced order:

$$C_0(s) = \frac{c_{n0} + c_{n1}s + \dots + c_{np}s^p}{c_{d0} + c_{d1}s + \dots + c_{dv}s^v}$$

where $v \geq p$ is less than the order of the system (plus weightings)

Lead lag:

$$C_0(s) = \frac{(c_{n0} + c_{n1}s)(c_{n2} + c_{n3}s)}{(c_{d0} + c_{d1}s)(c_{d2} + c_{d3}s)}$$

PID:

$$C_0(s) = k_0 + k_1 / s + k_2 s$$

The assumption must be made that a stabilising control law exists for the assumed controller structure. Note that the controller structure should be consistent with the choice of error weighting, in the sense that, if A_q includes a j axis zero, then the controller denominator $C_{0d}(s)$ should also include such a zero. The solution of this optimisation problem may be obtained using the following results. Assume, for example, that C_0 has a modified PID structure of the form:

$$C_0 = k_0 + (k_1 / s) + (k_2 s / (1 + s\tau)) \quad (4.12)$$

so that the numerator :

$$C_{0n} = k_0(1 + s\tau)s + k_1(1 + s\tau) + k_2 s^2 \quad (4.13)$$

and the denominator :

$$C_{0d} = s(1 + s\tau) \quad (4.14)$$

Let the superscripts r and i denote the real and imaginary parts of a complex function, so that $C_{0n} = C_{0n}^r + jC_{0n}^i$ and $C_{0d} = C_{0d}^r + jC_{0d}^i$. The controller numerator term may be split into frequency dependent components, through comparison with (4.13):

$$C_{0n}(j\omega) = -k_0\omega^2\tau + k_1 - k_2\omega^2 + j(k_0\omega + k_1\omega\tau) \quad (4.15)$$

and

$$C_{0n}^r(j\omega) = -k_0\omega^2\tau + k_1 - k_2\omega^2 \quad \text{and} \quad C_{0n}^i(j\omega) = k_0\omega + k_1\omega\tau \quad (4.16)$$

Similarly, for the denominator term:

$$C_{0d}(j\omega) = -\omega^2\tau + j\omega \quad (4.17)$$

and hence

$$C_{0d}^r(j\omega) = -\omega^2\tau \quad \text{and} \quad C_{0d}^i(j\omega) = \omega \quad (4.18)$$

If the solution of the optimization problem is to be found by iteration, the denominator term in T_l^+ can be assumed to be known and the minimisation can then be performed on the numerator (linear terms). Thus, to set up this problem let,

$$T_l^+ = C_{0n}L_{n1} - C_{0d}L_{n2}$$

where

$$L_{n1} = L_1 / (C_{0n}L_3 + C_{0d}L_4) \quad \text{and} \quad L_{n2} = L_2 / (C_{0n}L_3 + C_{0d}L_4) \quad (4.19)$$

Substituting from (4.11) and (4.19).

$$T_1^+ = C_{0n}^r L_{n1}^r - C_{0n}^i L_{n1}^i - C_{0d}^r L_{n2}^r + C_{0d}^i L_{n2}^i + j(C_{0n}^i L_{n1}^r + C_{0n}^r L_{n1}^i - C_{0d}^r L_{n2}^i - C_{0d}^i L_{n2}^r)$$

and after substitution from (4.16) and (4.18) obtain

$$T_1^+ = \left\{ \begin{aligned} &k_0 \left(-\omega^2 \tau L_{n1}^r - \omega L_{n1}^i + j(\omega L_{n1}^r - \omega^2 \tau L_{n1}^i) \right) + k_1 \left(L_{n1}^r - \omega \tau L_{n1}^i + j(\omega \tau L_{n1}^r + L_{n1}^i) \right) \\ &+ k_2 \left(-\omega^2 L_{n1}^r - j\omega^2 L_{n1}^i \right) + \omega^2 \tau L_{n2}^r + \omega L_{n2}^i + j(\omega^2 \tau L_{n2}^i - \omega L_{n2}^r) \end{aligned} \right\}$$

The real and imaginary part of T_l^+ may therefore be written as: $T_l^+ = T_l^{+r} + jT_l^{+i}$ and it follows that,

$$\left| T_l^+ \right|^2 = \left(T_l^{+r} \right)^2 + \left(T_l^{+i} \right)^2$$

Write a vector form of the above equations as:

$$\begin{bmatrix} T_l^{+r} \\ T_l^{+i} \end{bmatrix} = F \begin{bmatrix} k_0 \\ k_1 \\ k_2 \end{bmatrix} - L = Fx - L$$

where

$$F(\omega) = \begin{bmatrix} -\omega(\omega \tau L_{n1}^r + L_{n1}^i) & (L_{n1}^r - \omega \tau L_{n1}^i) & -\omega^2 L_{n1}^r \\ \omega(L_{n1}^r - \omega \tau L_{n1}^i) & (\omega \tau L_{n1}^r + L_{n1}^i) & -\omega^2 L_{n1}^i \end{bmatrix} \text{ and } L(\omega) = \begin{bmatrix} -\omega(L_{n2}^i + \omega \tau L_{n2}^r) \\ \omega L_{n2}^r - \omega^2 \tau L_{n2}^i \end{bmatrix}$$

The cost-function can be optimised directly but a simple iterative solution can be obtained if the integral is approximated (*Yukitomo et al* 1998 [23]) by a summation with a sufficient number of frequency points $\{\omega_1, \omega_2, \dots, \omega_N\}$. The optimisation can then be performed by minimising the sum of squares at each of the frequency points. The minimization of the cost term J_0 is therefore required where,

$$J_0 = \sum_{k=1}^N (Fx - L)^T (Fx - L) = (b - Ax)^T (b - Ax) \quad (4.20)$$

where

$$A = \begin{bmatrix} F(\omega_1) \\ \vdots \\ F(\omega_N) \end{bmatrix}, \quad b = \begin{bmatrix} L(\omega_1) \\ \vdots \\ L(\omega_N) \end{bmatrix}, \quad x = \begin{bmatrix} k_0 \\ k_1 \\ k_2 \end{bmatrix} \quad (4.21)$$

Assuming the matrix $A^T A$ is non-singular the least squares optimal solution (*Noble* 1969 [24]) follows as:

$$x = (A^T A)^{-1} A^T b \quad (4.22)$$

Lemma 4.1: Restricted Structure LQG Controller Solution Properties

The *characteristic polynomial* which determines stability and the *implied equation* are given as:

$$\begin{aligned} \rho_c &= AC_{0d} + BC_{0n} \\ AA_q H_0 + BA_r G_0 &= D_c D_f \end{aligned} \quad (4.23)$$

The minimum value of the cost-function, with controller of restricted structure, is given as:

$$J_{\min} = \frac{1}{2\pi j} \oint_D \{T_1^+ T_1^{+*} + T_1^- T_1^{-*} + \Phi_0\} ds$$

$$J_{\min} = \frac{1}{2\pi j} \oint_D \{T_1^+ T_1^{+*} + \frac{(Y_0 - F_0 D_f^*)^* (Y_0 - F_0 D_f^*)}{D_c^* D_c D_f D_f^*} + \Phi_0\} ds \quad (4.24)$$

Proof: The implied equation (4.23) follows from (4.7) and (4.8) by multiplying (4.7) by BA_r and (4.8) by AA_q , and adding.

4.1 Design and Robustness Improvement

The LQG controller should be designed in such a way that it is consistent with the restricted controller structure of interest. For example, A_q should approximate a differentiator if near integral action is required. In fact the assumption made in deriving Theorem 3.1 was that the controller structure is compatible with the choice of error weighting and if $1/A_q$ includes a j axis pole then this will be included in the chosen controller. In fact the usual situation will be that the designer decides the controller should include integral action and the weighting ($1/A_q$) will be chosen as an integrator. The control weighting $1/A_r$ is not so critical but if for example, a PID structure is to be used, then the point at which the differential (lead term) comes in can help to determine the A_r weighting. Clearly, there is no point in designing an LQG controller which has an ideal response, in some sense, but cannot be approximated by the chosen controller structure. Thus, the weightings should be selected so that the closed-loop properties are satisfactory but taking into consideration the limitations of the controller structure required. The basic concept proposed is straightforward. That is, in the region of the unity gain crossover frequency for the open loop system, or the phase margin frequency, the distance $|1+WC_0|$ should normally be maximised. This requires the sensitivity to be minimized, particularly in this sensitive region. By costing the sensitivity directly a mechanism is provided to improve robustness (Horowitz 1979 [18]) but there are some subtleties to address:

1. The weighting P_c needs to be increased from zero where performance is presumably maximized (the LQG cost is optimized) up to a level where robustness is adequate and performance still acceptable.
2. If pure sensitivity costing is required P_c could cancel the combined noise dynamics $\Phi_{ff} = Y_f Y_f^*$ which is unrealistic. The alternative is to make the model $W_p = A^{-1} C_p$ large, relative to the other noise terms and also a constant ($C_p = \rho A$ say) and this will introduce a fictitious stochastic term affecting the noise and disturbance rejection, and reference tracking, properties. The size of the scalar ρ also therefore involves a compromise.
3. The weighting and frequency shaping effects are only important in the decade above and a little below the crossover frequency referred to. The shaping might therefore be introduced by a weighting that approximates an ideal window function but this increases the order of the weighting term.

The above design choices and trade-offs detract from the approach but the prize is quite important and worthy of the effort. That is, the provision of a tuning variable, or variables, in a cost index where the robustness properties of an optimal controller can be manipulated and traded against performance/stochastic properties.

5.0 COST-FUNCTION EVALUATION AND CONTROLLER BENCHMARK COST COMPUTATION

The minimisation of the cost-function can be shown to be equivalent to the minimisation of the variance of the adjoint output signal (see Figure 2).

$$\phi = D_c^{*-1} (A_1^* R_{cn} A_r^{-1} u - B_1^* \bar{Q}_{cn} A_q^{-1} e_0) \quad (5.1)$$

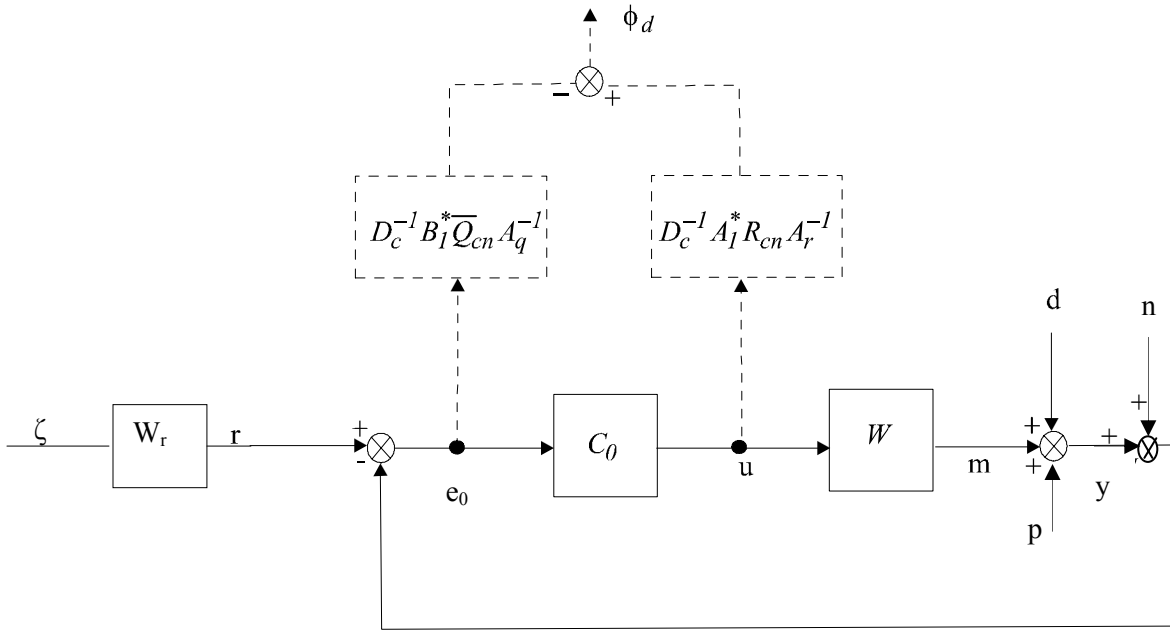


Figure 2: Block Diagram representation of Benchmarking Cost.

The polynomial operator version of the control spectral factor may be defined using:

$$Y_c^* Y_c = W^* \tilde{Q}_c W + R_c = A_c^{*-1} (B_1^* \bar{Q}_{cn} B_1 + A_1^* R_{cn} A_1) A_c^{-1}$$

where D_c is a polynomial operator that satisfies:

$$D_c^* D_c = B_1^* \bar{Q}_{cn} B_1 + A_1^* R_{cn} A_1$$

and Y_c may be written as: $Y_c = D_c A_c^{-1}$ where $A_I = A A_q$ and $B_I = B A_r$, and $A_c = A A_q A_r$. Assume the existence of the solutions (G_0, F_0) , (H_0, F_0) of the equations (4.7) and (4.8). These equations can be added, after first multiplying by $D_f^{-1} B_I$ and $D_f^{-1} A_I$, respectively, to obtain:

$$\begin{aligned} D_c^* (G_0 D_f^{-1} B_l + H_0 D_f^{-1} A_l) + F_0 (A A_q D_f^{-1} B_l - B A_r D_f^{-1} A_l) \\ = (B_l^* \bar{Q}_{cn} B_l + A_l^* R_{cn} A_l) \end{aligned} \quad (5.2)$$

but the last two of the terms in (5.2) add to zero. The implied diophantine equation then follows from (5.2) by dividing by D_c^* :

$$G_0 B_l + H_0 A_l = D_c D_f \quad (5.3)$$

The variance of such a signal $\phi = D_c^{*-1} (A_l^* R_{cn} A_r^{-1} u - B_l^* \bar{Q}_{cn} A_q^{-1} e_0)$ can be shown (using Parseval's theorem) to be the same as the signal (Grimble 1984 [18]):

$$\phi_d = D_c^{-1} D_c^* \phi = D_c^{-1} (A_l^* R_{cn} A_r^{-1} u - B_l^* \bar{Q}_{cn} A_q^{-1} e_0)$$

The solution of the LQG optimal control problem therefore involves the minimization of the signal shown ϕ_d in Fig. 2. This is valuable in providing a link to the benchmarking techniques for minimum variance controllers that are well established. The diophantine equation (4.7) may be written in the form:

$$\frac{D_c^*}{D_c} \frac{G_0}{A A_q} + \frac{F_0}{D_c} = D_c^{-1} B_l^* \bar{Q}_{cn} A_q^{-1} \cdot A^{-1} D_f$$

Recalling that $f = A^{-1} D_f \in$ the equation above may be used to obtain:

$$\frac{D_c^*}{D_c} \frac{G_0}{A A_q} \in + \frac{F_0}{D_c} \in = D_c^{-1} B_l^* \bar{Q}_{cn} A_q^{-1} \cdot f$$

From this last result one interpretation of the term $F_0 D_c^{-1} \in$, which has the same variance as part of the cost term J_l , (defined in (5.6)) is one component of f in the signal ϕ_d in Fig. 2. In minimum variance benchmarking problems this term is of the simpler form $F_0 \in$ and the coefficients of F_0 are available from simple test results involving the signal f (considered in the discrete time case by Huang and Shah 1999 [30]).

The value of the cost-function and of the error and control signal variances can be evaluated using the solution of the equations in Theorem 3.1. The following theorem provides the expressions for the cost values, for both the case of any (suboptimal) controller and for the optimal solution.

Theorem 3.2:

The performance criterion (3.1) may be evaluated for any linear controller $C_0 = C_{0d}^{-1} C_{0n}$ as:

$$J = J_0 + J_1 + J_2 \quad (5.4)$$

where

$$J_0 = \frac{1}{2\pi j} \oint_D \{T_1^+ T_1^{+*}\} ds \quad (5.5)$$

$$J_1 = \frac{1}{2\pi j_D} \oint \left\{ \left(\frac{Y_0 - F_0 D_f^*}{D_f^* D_c^*} \right) \left(\frac{Y_0 - F_0 D_f^*}{D_f^* D_c^*} \right)^* \right\} ds \quad (5.6)$$

$$J_2 = \frac{1}{2\pi j_D} \oint \left\{ \bar{Q}_c \Phi_{ff} - Q_c \Phi_{nn} - \frac{\Phi_{fp} \Phi_{fp}^*}{Y_c Y_c^* Y_f Y_f^*} \right\} ds \quad (5.7)$$

where

$$T_1^+ = \frac{[(H_0 A_{n0} A_q + X_0 A_r B) C_{0n} - (G_0 A_{n0} - X_0 A) A_r C_{0d}]}{A_{n0} A_q A_r (A C_{0d} + B C_{0n})} \quad (5.8)$$

The variances of the error and control signals may also be computed, for any controller, as:

$$J_e = \frac{1}{2\pi j_D} \oint (T_e T_e^*) ds + \frac{1}{2\pi j_D} \oint \{ \Phi_{nn} - (S + S^*) \Phi_{nn} \} ds \quad (5.9)$$

$$J_u = \frac{1}{2\pi j_D} \oint \{ T_u T_u^* \} ds \quad (5.10)$$

where

$$T_e = C_{0d} (A C_{0d} + B C_{0n})^{-1} D_f \quad \text{and} \quad T_u = C_0 T_e \quad (5.11)$$

The optimal value of the cost term J_0 , for the full order optimal controller, is $J_{0min} = 0$. The minimum-cost in the full-order optimum case can be obtained in the alternative forms:

$$J_{min} = J_1 + J_2 \quad (5.12)$$

$$J_{min} = J_{qmin} + J_{rmin} + J_{pmin} \quad (5.13)$$

where

$$J_{qmin} = \frac{1}{2\pi j_D} \oint \{ S_{min} Y_f Q_c Y_f^* S_{min}^* + \Phi_{nn} Q_c - (S_{min} + S_{min}^*) Q_c \Phi_{nn} \} ds \quad (5.14)$$

$$J_{rmin} = \frac{1}{2\pi j_D} \oint \{ M_{min} Y_f R_c Y_f^* M_{min}^* \} ds = \frac{1}{2\pi j_D} \oint \{ T_{rmin} T_{rmin}^* \} ds \quad (5.15)$$

$$J_{pmin} = \frac{1}{2\pi j_D} \oint \{ S_{min} Y_f P_c Y_f^* S_{min}^* \} ds = \frac{1}{2\pi j_D} \oint \{ T_{pmin} T_{pmin}^* \} ds \quad (5.16)$$

and

$$T_{rmin} = \frac{B_r (G_0 A_{n0} - X_0 A)}{D_c U_0}, \quad T_{pmin} = \frac{B_p (H_0 A_{n0} A_q - X_0 A_r B)}{A_q D_c U_0} \quad (5.17)$$

The optimal values of the variances for the error and control signals for the full-order optimal controller follow as:

$$J_{e\min} = \frac{1}{2\pi j_D} \oint \left\{ T_{e\min} T_{e\min}^* + \Phi_{nn} - (S_{\min} + S_{\min}^*) \Phi_{nn} \right\} ds \quad (5.18)$$

$$J_{u\min} = \frac{1}{2\pi j_D} \oint \left\{ M_{\min} Y_f Y_f^* M_{\min}^* \right\} ds = \frac{1}{2\pi j_D} \oint \left\{ T_{u\min} T_{u\min}^* \right\} ds \quad (5.19)$$

where

$$S_{\min} = \frac{A_0 (H_0 A_{n0} A_q + X_0 A_r B)}{D_c D_f} \quad (5.20)$$

$$M_{\min} = \frac{A_0 A_r (G_0 A_{n0} - X_0 A)}{D_c D_f} \quad (5.21)$$

and

$$T_{u\min} = \frac{A_r (G_0 A_{n0} - X_0 A)}{D_c U_0}, \quad T_{e\min} = \frac{H_0 A_{n0} A_q - X_0 A_r B}{D_c U_0} \quad (5.22)$$

The existing system might have a classically designed controller, of more conventional structure (like a PID controller), and although it is of some value to compare this cost J for this classical controller with the best that can be achieved ($J_{\min} = J_1 + J_2$), this is often an unfair benchmark comparison. The reason is of course that the restriction on the controller structure and the order, may make the absolute minimum J_{\min} way below what is achievable using the best tuned classical controller. A more appropriate benchmark figure is therefore to compare the actual cost with that of the optimal cost assuming the controller structure is fixed. If the controller has a specified limited structure, the minimum of the cost term $J_{0\min}$ will be non-zero. The minimum value of the criterion (3.1) for this restricted controller will be obtained as $J_{\text{restrict}} = J_{0\min} + J_{\min}$. For an unconstrained solution the minimum is achieved when $T_1^+ = 0$ and the minimum of J_0 (denoted $J_{0\min}$) is zero. The increase in cost which occurs by restricting the controller structure, can be obtained as,

$$\Delta J_{\min} = \frac{1}{2\pi j_D} \oint (T_1^+ T_1^{+*}) ds \quad (5.23)$$

The Controller Performance Index will be defined as the ratio of the minimum possible value of the cost function (3.1) to the actual value and the CPI lies between $0 \leq \kappa \leq 1$. If κ is close to unity the system provides little opportunity for improvement. If the CPI is close to zero retuning is recommended. This scalar is similar but not the same, as the assessment measure introduced by Desborough and Harris (1992 [29])

$$\kappa = J_{\min} / (J_{\min} + J_0) = 1 - J_0 / (J_{\min} + J_0) \quad (5.24)$$

6.0 ROBUST CONTROL DESIGN EXAMPLE

To illustrate the effect of the robustness weighting element and the fictitious robustness signal $\{p(t)\}$ a simple example is considered. The system models may be listed as:

Coprime system model:

$$W = \frac{B_0}{A_0} = \frac{1000(s+2)(s+6)}{(s+0.7)(s+3.9)(s+100)(s^2 + 2\xi\omega_0s + \omega_0^2)}$$

where $\xi = 0.1$ and $\omega_0 = 10$. Also write

$$W = \frac{B}{A} = \frac{B_0 U_0}{A_0 U_0} \quad \text{where} \quad U_0 = (s+3.2)s$$

Disturbance model:

$$W_d = \frac{C_d}{A} = \frac{1000}{(s+100)(s^2 + 2\xi\omega_0s + \omega_0^2)(s+3.2)s}$$

where

$$C_d = 1000(s+0.7)(s+3.9)$$

Reference model:

$$W_r = \frac{E}{A} = \frac{1}{(s^2 + 2\xi\omega_0s + \omega_0^2)s}$$

where

$$E = (s+0.7)(s+3.9)(s+3.2)(s+100)$$

Noise model:

$$W_n = \frac{C_n}{A} = \frac{0.1}{(s+100)}$$

where

$$C_n = 0.1(s+0.7)(s+3.9)(s^2 + 2\xi\omega_0s + \omega_0^2)(s+3.2)s$$

Fictitious robustness signal:

$$W_p = \frac{C_p}{A} = \frac{\rho}{(s+100)}$$

Cost Function Weightings

The cost function weightings may be defined as:

$$Q_c = \frac{(0.01s+1)(-0.01s+1)}{(s+10^{-6})(-s+10^{-6})}$$

$$R_c = (10s+1)(-10s+1)$$

$$P_c = \rho_1 \frac{(0.01s+1)(-0.01s+1)}{(s+10^{-6})(-s+10^{-6})}$$

Results

The frequency responses of the different system models are shown in Fig. 3. The system is low pass with a resonant subsystem and the disturbance model includes an integrator. The measurement noise model only rolls off at high frequencies. Consider first the full order optimal case and the use of a large $\rho = 1000$, then as ρ_1 varies the unit step responses of the closed loop system are as shown in Fig. 4. This represents the case where there is a large fictitious disturbance model W_p but where the robustness weighting ρ varies between 0 to 1000. The faster responses occur as ρ increases, since the effect is related to that when Q_c increases. The corresponding closed-loop frequency responses are shown in Fig. 4.

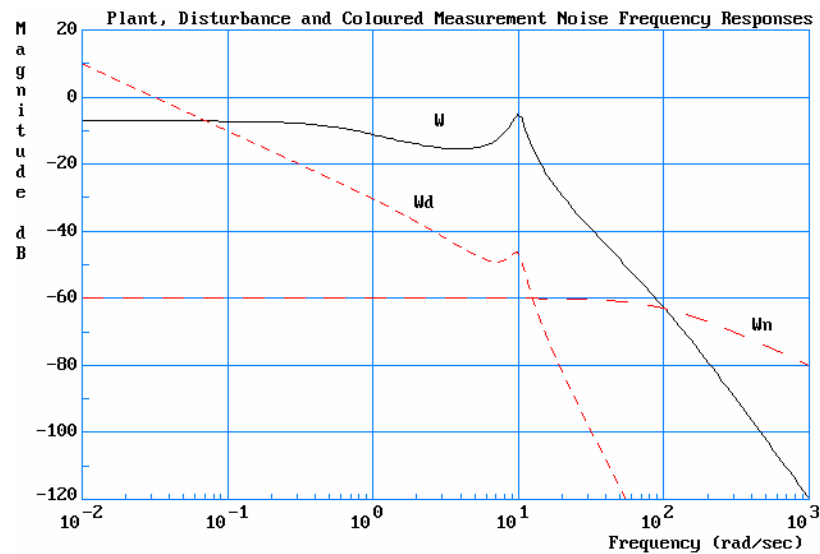


Figure 3: Bode Frequency Responses for the System, Disturbance and Noise.

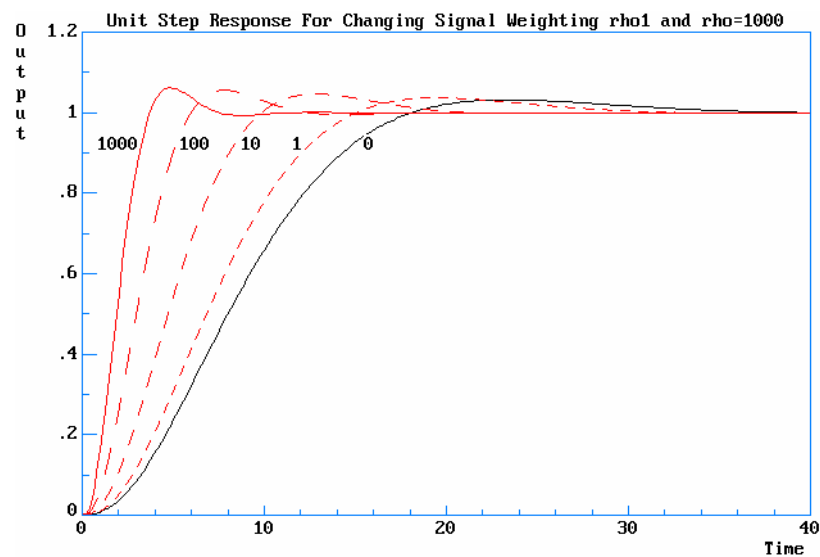


Figure 4: Closed Loop System Unit Step Responses for Fixed $\rho = 1000$ and Varying Weighting ρ_1 .

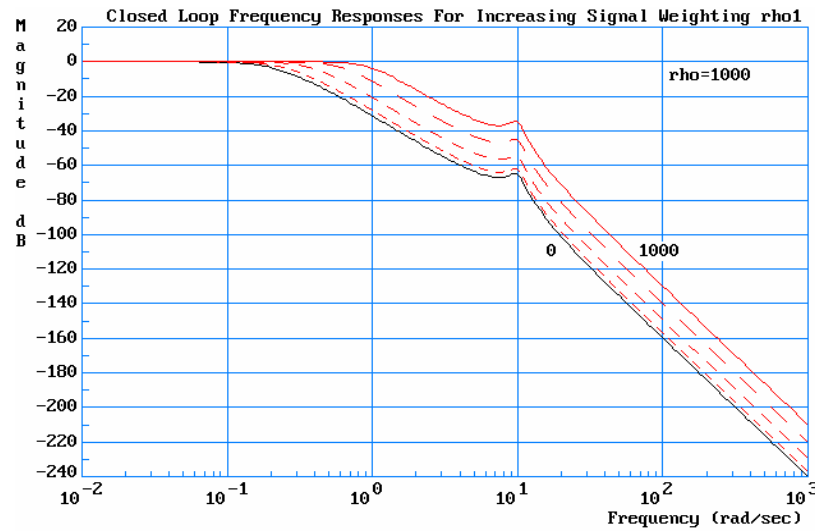


Figure 5: Bode Diagram of Closed Loop Responses for $\rho = 1000$ and Changing Value of ρ_1 .

Restricted Structure

The unit step responses are compared in Fig. 6 for the full-order and restricted controller designs. The case $\rho = 0$ and $\rho_1 = 10$ was considered and the corresponding controller and sensitivity-function frequency responses are shown in Figs 7 and 8. The restricted structure control is particularly good from an overshoot perspective. However, one reason is the higher controller gains at high frequencies for the restricted structure control law. The computed controllers were obtained as:

Optimal Full-Order Controller:

$$C_0(s) = \frac{0.493426(s + 0.2205695)(s + 0.7)(s + 3.199631)(s + 3.9)(s^2 + 1.999939s + 99.99969)(s + 100)}{(s + 0.000001)(s + 1.09342 \times 10^{-6})(s + 1.008371)(s + 3.866217)(s^2 + 14.81793s + 65.05414) \times (s^2 + 7.520861s + 166.0202)}$$

Optimal Restricted Structure:

$$C_0(s) = \frac{-0.0641248s^2 - 0.3587621s + 1.208584}{s(0.2s + 1)}$$

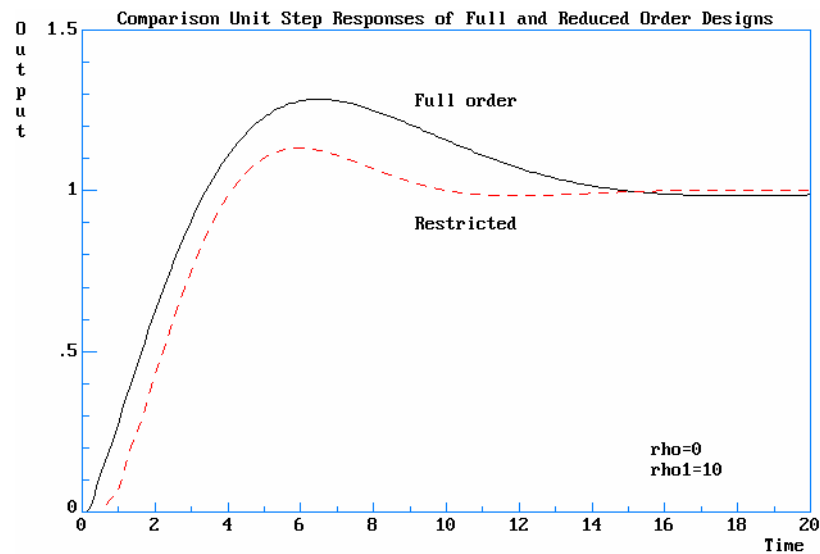


Figure 6: Comparison of Closed-Loop Unit Step Responses of Full and Restricted Structure Control Designs.

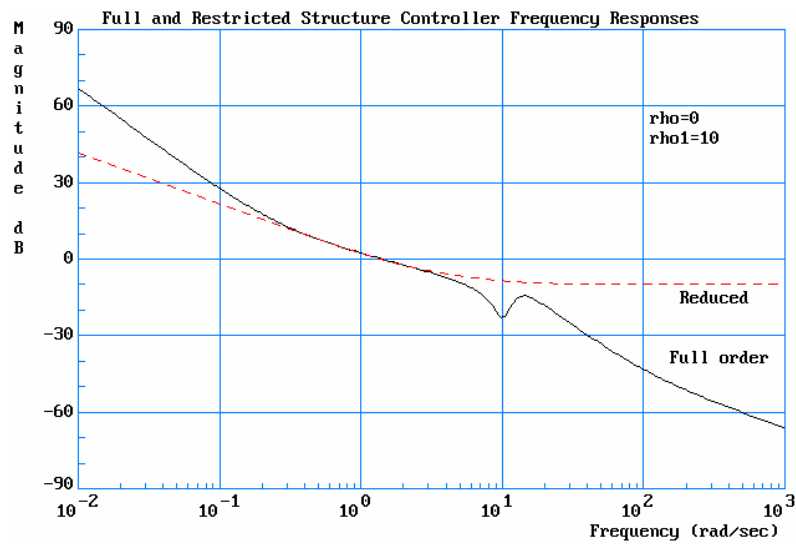


Figure 7: Bode Comparison of Controller Frequency Responses.

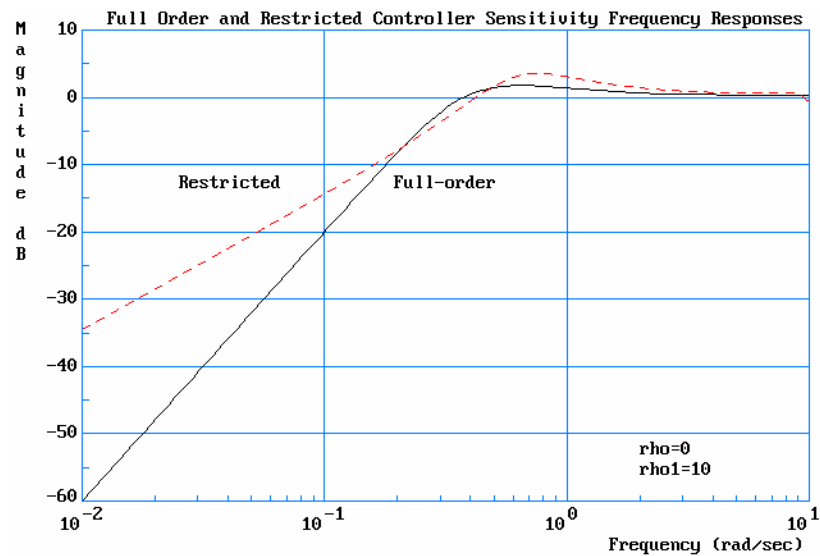


Figure 8: Bode Comparison of Sensitivity Function Frequency Responses.

The step responses shown in Fig. 9 are for the case $\rho = \rho_1 = 100$. The results are much faster and the restricted structure design is again good, relative to the full-order solution.

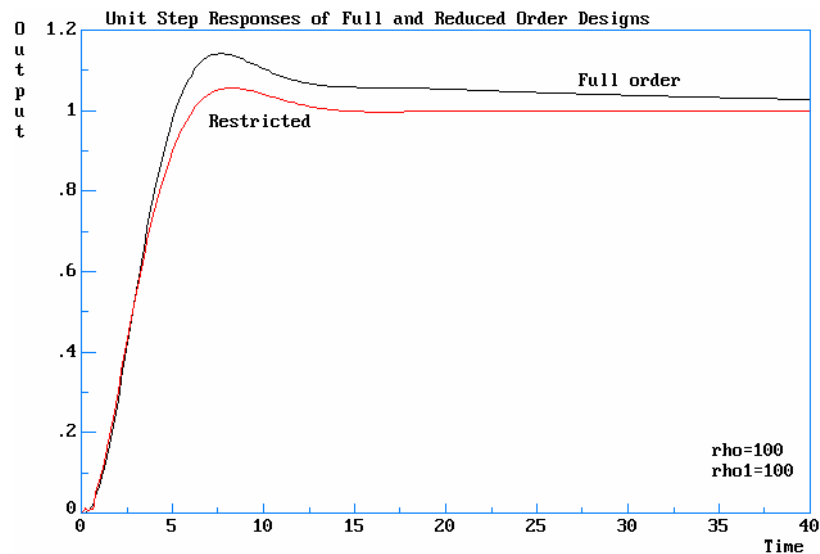


Figure 9: Comparison of Unit Step Responses for Full-Order and Restricted Structure Control Designs.

Example Conclusions

The example reveals that the robustness weighting terms and the fictitious robust costing signal $\{p(t)\}$ certainly affects the overshoots which represent a measure of robustness, both on closed-loop frequency and time responses. The tuning variables ρ and ρ_1 affect the robustness of this minimum-phase open loop stable system in much the way expected. However, the alteration of robustness properties is not a straightforward matter, since any values of ρ and ρ_1 above zero will cause a measure of sub-optimality in stochastic (LQG cost) terms. The most surprising results were the very good results obtained for the restricted structure control designs. The explanation was the higher high frequency gains employed that reduced the peaks on the sensitivity function frequency responses. In this problem changes in the measurement noise model again did not have a large effect. However, results that were not shown were obtained for a coloured measurement noise model with a peak in the low frequency range. In this case the controller gains are significantly reduced and this slows the speed of response of the system and the overshoot increases markedly.

7.0 CONCLUSIONS

The robustness of full-order and restricted structure optimal control problems was considered for continuous-time linear systems. The emphasis was on the improvement of robustness by adding a sensitivity costing term in the cost index and by introducing a fictitious disturbance model W_p . The effect of measurement noise was also investigated and this was introduced in the feedback system model. The robustness weighting acts directly on the sensitivity function and may improve robustness margins but other properties will probably deteriorate like the measurement noise rejection properties. If robustness is more important than stochastic properties then attention would turn to H_∞ cost minimization and many of the ideas presented above would apply (Grimble, 1986 [20]). However, such an approach is readily embedded in the usual mixed sensitivity H_∞ design problem.

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